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ON WAVE EXCITATION BY A VIBRATING STAMP IN A MEDIUM WITH INHOMOGENEOUS INITIAL STRESSES*

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One possible means for studying the features of excitation of an inhomogeneous medium with inhomogeneous initial stresses by an oscillating stamp is proposed (the mechanical parameters of the material of the medium and the initial stress tensor components are arbitrary, sufficiently smooth functions of one of the coordinates, the depth of the layer, the radius of the cylinder). The approach being developed to study inhomogeneous prestresses media is realized in the solution of the problem of the oscillating of a stamp on a lyer surface and the problem of the vibration of a stiff belt on the surface of an infinite circular cylinder. Integral equations are derived to whose study the solution of both problems reduces, and the properties of the kernels are investigated. The single-valued solvability of the integral equations in a certain class of functions is shown. The influence of the properties of the material of the medium and of the nature of the change in the initial stresses on the contact stress distribution under the stamp, and on the behavior of the free surface outside it is investigated numerically.

Analogous problems are examined in /1,2/ on the basis of a linearized theory of elastic wave propagation /3/ under the assumption of initial isotropy of the medium and homogeneity of the initial stresses.

1. The problem of exciting an elastic medium with initial strains (in the absence of mass forces) is described by the relationships /4/

$$\nabla\cdot\Theta=\rho\mathbf{u}^{\cdot\cdot},\quad \mathbf{N}\cdot\Theta=\mathbf{q};\quad \Theta=\mathbf{P}+{}^1/_2(\mathbf{T}\cdot\varepsilon-\varepsilon\cdot\mathbf{T})-\mathbf{T}\cdot\Omega \tag{1.1}$$

Here ρ is the density of the material in the medium $\mathbf{u}=\{u_1,u_2,u_3\}$, $\mathbf{q}=\{q_1,q_2,q_3\}$ and $\mathbf{N}=\{N_1,N_2,N_3\}$ are, respectively, the displacement, surface force, and normal vectors to the surface. The initial stress tensor $\mathbf{T}$ and the symmetric ε and skew-symmetric Ω strain tensors take part in representing the tensor of second rank Θ. The symmetric tensor $\mathbf{P}$ depends only on the properties of the material and in the case of small initial strain can be represented in the form /5/

$$\mathbf{P}(\varepsilon)=\lambda\,\mathrm{tr}\,\varepsilon\cdot\mathbf{E}+2\mu\varepsilon$$

Here $\mathbf{E}$ is the unit tensor, and λ and μ are Lamé parameters.

We rewrite the equation and boundary condition (1.1) in the form

$$\frac{\partial}{\partial x_k}\Theta_{ks}=\rho u_s^{\cdot\cdot},\quad k,s=1,2,3 \tag{1.2}$$

$$N_k\Theta_{ks}=q_s,\quad \Theta_{ks}=\tau_{ks}+{}^1/_2(\sigma_{kn}\varepsilon_{ns}-\varepsilon_{nk}\sigma_{sn})-\sigma_{kn}\Omega_{ns} \tag{1.3}$$

$$\tau_{ks}=\lambda\delta_{ks}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+2\mu\varepsilon_{ks}$$

Here σ_{ij} are the initial stress tensor components, ε_{ks}, Ω_{ks} are the symmetric and skew-symmetric strain tensor components, τ_{ks} are the stress tensor components related to the strain tensor components by the above-mentioned dependence.

Assuming $\sigma_{ij}=0$, $i\neq j$, $i,j=1,2,3$, we obtain the following form for the components of the tensor Θ:

$$\Theta_{ks}=a_{ks}\frac{\partial u_k}{\partial x_s}+b_{ks}\frac{\partial u_s}{\partial x_k},\quad k\neq s \tag{1.4}$$

$$\Theta_{kk}=(\lambda+2\mu)\frac{\partial u_k}{\partial x_k}+\lambda\left(\frac{\partial u_i}{\partial x_i}+\frac{\partial u_j}{\partial x_j}\right),\quad k\neq i,\quad k\neq j$$

$$a_{ks}=\mu(1-\sigma_k-\sigma_s),\quad b_{ks}=\mu(1+3\sigma_k-\sigma_s),\quad k\neq s$$

$$\sigma_k=\sigma_{kk}/(4\mu);\qquad i,j,k,s=1,2,3$$

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2. Let us consider the plane problem of the vibration of a rigid stamp of width $2a$ on the surface of a layer occupying the domain $|x_1|, |x_2| \leqslant \infty, 0 \leqslant x_3 \leqslant h$. Assuming the coefficients $\lambda, \mu, a_{ks}, b_{ks}$ in the representation (1.4) to be functions of x_3, and applying the Fourier transform in x_1 to (1.2) (α is the transformation parameter, and U_1, U_3 are Fourier transforms of the functions u_1, u_3), by using the notation $U_1' = y_1, -i\alpha U_3' = y_2, U_1 = y_3, -i\alpha U_3 = y_4$ we arrive at the system

$$y_1' = b_{31}^{-1}\{-b_{31}'y_1 + [(\lambda + 2\mu)\alpha^2 - \rho\omega^2]y_3 - a_{31}'y_4 - (\lambda + a_{31})y_2\} \tag{2.1}$$

$$y_2' = (\lambda + 2\mu)^{-1}[\alpha^2(\lambda + a_{13})y_1 - (\lambda' + 2\mu')y_2 + \alpha^2\lambda' y_3 + (\alpha^2 b_{13} - \rho\omega^2)y_4]$$

$$y_3' = y_1, \; y_4' = y_2$$

Proceeding in an analogous manner, we reduce the boundary conditions (1.3) to the form

$$x_3 = h, \; b_{31}y_1 + a_{31}y_4 = 0, \; -\alpha^2\lambda y_3 + (\lambda + 2\mu)y_2 = Q(\alpha) \tag{2.2}$$
$$x_3 = 0, \; y_3 = y_4 = 0$$

($Q(\alpha) = -i\alpha T_3(\alpha)$ is the Fourier transform of $q_3(x_1)$).

For a further analysis it is necessary to have four linearly independent solutions of the system (2.1). They can be obtained numerically by the Runge—Kutta, Adams, etc. methods, say /6,7/. Let us assume that these solutions with the initial conditions $y_{ik}(0) = \delta_{ik}$ are constructed and have the form

$$y_i = \sum_{k=1}^{4} c_k y_{ik}(x_3), \quad i, k = 1, 2, 3, 4 \tag{2.3}$$

Then the solution of the problem under consideration can be written in the form

$$u_i(x_1, x_3) = \frac{1}{2\pi}\int_{-a}^{a} k(x_3, \xi - x_1)q(\xi)\,d\xi, \quad i = 1, 3 \tag{2.4}$$

$$k(x_3, t) = \int_\Gamma K(\alpha, x_3)e^{i\alpha t}\,d\alpha \tag{2.5}$$

$$K(\alpha, x_3) = [A_1 y_{i1}(x_3) - A_2 y_{i2}(x_3)]/\Delta \tag{2.6}$$
$$A_k = b_{31}(h)y_{1k}(h) + a_{31}(h)y_{4k}(h), \; k = 1, 2$$
$$\Delta = [\lambda(h) + 2\mu(h)][y_{22}(h)A_1 - y_{21}(h)A_2] -$$
$$\alpha^2\lambda(h)[y_{32}(h)A_1 - y_{31}(h)A_2]$$

The right side of (2.4) governs the displacement of an arbitrary point of the layer subjected to the load $q_3(x_1)$ given in $[-a, a]$. Setting $x_3 = h$ in (2.4)—(2.6), we obtain the layer surface displacement determined by the relationships

$$u_i^\circ(x_1) = \frac{1}{2\pi}\int_{-a}^{a} k^\circ(\xi - x_1)q(\xi)\,d\xi \tag{2.7}$$

$$k^\circ(t) = \int_\Gamma K^\circ(\alpha)e^{i\alpha t}\,d\alpha, \quad K^\circ(\alpha) = K(\alpha, h) \tag{2.8}$$

The contour Γ in the representations (2.5) and (2.8) is selected in conformity with rules elucidated in /8/, except after a numerical analysis of the properties of the function $K^\circ(\alpha)$. In the case of the problem on the action of a stamp on a layer surface, the relationship (2.7) is an integral equation in the unknown function $q(\xi)$.

3. The evenness and meromorphism of the function $K^\circ(\alpha)$ are determined from the form of the analytic dependence of the coefficients of the differential equations (2.1) on the parameter α. The presence of real zeros and poles and the nature of their distribution can be determined only upon having the specific form of the functions $\lambda(x_3), \mu(x_3), \rho(x_3), \sigma(x_3)$ given. The asymptotic behavior of $K^\circ(\alpha)$ as $\alpha \to \infty$ plays an important role.

The kernel of the integral equation is constructed numerically as the solution of the boundary value problem (2.1), (2.2); hance it is natural to identify the asymptotic of the kernel with the asymptotic of the corresponding boundary operator. By using the notation

$$g_1 = -i\varepsilon^2 y_2, \; g_2 = i\varepsilon y_1, \; q_3 = i\varepsilon y_4, \; g_4 = -iy_3$$
$$\varepsilon = \alpha^{-1}, \; \alpha_1 = -(\lambda + a_{13})/(\lambda + 2\mu), \; \alpha_2 = -b_{13}/(\lambda + 2\mu)$$
$$\alpha_3 = (\lambda + a_{31})/b_{31}, \qquad \alpha_4 = -(\lambda + 2\mu)/b_{31}$$

we reduce the problem (2.1), (2.2) to the form

$$\varepsilon g_1' = \alpha_1 g_2 - (\lambda + 2\mu)^{-1} [(\lambda' + 2\mu') \varepsilon g_1 - (b_{13} - \rho\omega^2\varepsilon^2) g_3 - \varepsilon\lambda' g_4] \tag{3.1}$$

$$\varepsilon g_2' = \alpha_3 g_1 - b_{31}^{-1} [\varepsilon b_{31}' g_2 + ((\lambda + 2\mu) - \rho\omega^2\varepsilon^2) g_4 + \varepsilon a_{31}' g_3]$$

$$\varepsilon g_3' = -g_1, \ \varepsilon g_4' = -g_2$$

$$x_3 = h, \ b_{31} g_2 + a_{31} g_3 = 0, \ (\lambda + 2\mu) g_1 - \lambda g_4 = -\varepsilon \tag{3.2}$$

$$x_3 = 0, g_3 = g_4 = 0$$

The system (3.1) can be written in matrix form

$$\varepsilon \mathbf{G}' = \mathbf{A}(z, \varepsilon) \mathbf{G}, \ z \equiv x_3$$

Let us construct the asymptotic solution of this system. It follows from the form of (3.1) that the matrix $\mathbf{A}(z, \varepsilon)$ allows expansion in a power series in ε

$$\mathbf{A}(z, \varepsilon) \sim \sum_{k=0}^{\infty} \mathbf{A}_k(z) \varepsilon^k, \quad \varepsilon \to 0 \tag{3.3}$$

where

$$\mathbf{A}_0(z) = \begin{Vmatrix} 0 & \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & 0 & 0 & \alpha_4 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix} \tag{3.4}$$

The eigenvalues of the matrix $\mathbf{A}_0$

$$q_{1,2} = (-p \pm (p^2 - c)^{1/2})^{1/2}, \ q_{3,4} = -(-p \pm (p^2 - c)^{1/2})^{1/2}, \ (2p = -\alpha_1\alpha_3 + \alpha_2 + \alpha_4, \ c = b_{13}/b_{31})$$

are distinct. The asymptotic solution of the system (4.1) is written in this case in the form

$$g_j(z, \varepsilon) \sim \sum_{r=0}^{\infty} \varepsilon^r \left\{ \sum_{i=1}^{4} a_{jr}^{(i)}(z) \exp\left(\frac{1}{\varepsilon} \int_0^z q_i(t)\, dt\right) \right\} \tag{3.5}$$

where $a_{jr}^{(i)}$ is to be determined from (3.1) and (3.2). We shall substitute particular solutions corresponding to the distinct q_i alternately into (3.1) and equate expressions on the left and right for identical powers of ε. If the matrix

$$B^{(i)}(z) = \begin{Vmatrix} q_i & -\alpha_1 & -\alpha_2 & 0 \\ \alpha_3 & -q_i & 0 & \alpha_4 \\ 1 & 0 & q_i & 0 \\ 0 & 1 & 0 & q_i \end{Vmatrix}$$

is introduced into the consideration, then we obtain for $r = 0$ /9/

$$a_{j0}^{(i)} = c_i^\circ B_{kj}^{(i)} \tag{3.6}$$

Here $B_{kj}^{(i)}$ is the cofactor of the element b_{kj} of the matrix $B^{(i)}(z)$, and c_i° are constants to be determined. The left sides of (3.6) are determined by the formulas

$$a_{10}^{(i)} = -c_i^\circ (q_i^3 - \alpha_4 q_i), \quad a_{20}^{(i)} = c_i^\circ q_i^2 \alpha_4 \tag{3.7}$$

$$a_{30}^{(i)} = c_i^\circ (q_i^2 + \alpha_4), \ a_{40}^{(i)} = -c_i^\circ q_i \alpha_3$$

where the expressions in parentheses on the right are evidently not zero for all $a_{j0}^{(i)}$. The boundary conditions (3.2) are represented by the expressions

$$\sum_{i=1}^{4} c_i^\circ s_n^{(i)} = 0, \quad n = 1, 2, 3, 4 \tag{3.8}$$

$$s_1^{(i)} = \{b_{31} B_{12}^{(i)}(h) + a_{31} B_{13}^{(i)}(h)\} \exp\left(\frac{1}{\varepsilon} \int_0^h q_i(t)\, dt\right)$$

$$s_2^{(i)} = \{[\lambda(h) + 2\mu(h)] B_{11}^{(i)}(h) - \lambda(h) B_{14}^{(i)}(h)\} \exp\left(\frac{1}{\varepsilon} \int_0^h q_i(t)\, dt\right)$$

$$s_3^{(i)} = B_{13}^{(i)}(0), \ s_4^{(i)} = B_{14}^{(i)}(0)$$

Since $\det\|s_n^{(i)}\| \neq 0$, $i, n = 1, 2, 3, 4$, then it is necessary that $c_i^\circ = 0$, $i = 1, 2, 3, 4$, and therefore, $a_{j0}^{(i)} \equiv 0$. In this case the boundary conditions have the form

$$\sum_{i=1}^{4} c_1^{(i)} s_n^{(i)} = \begin{cases} -1, & n=2 \\ 0, & n\neq 2 \end{cases}, \quad \det\|s_n^{(i)}\| \neq 0$$

Then /9/

$$c_0^{(i)} = \frac{\det\|s_n^{(i)}\|_i}{\det\|s_n^{(i)}\|}$$

Taking into account (3.7), (3.5) and (3.8), it can be concluded that as $\varepsilon \to 0$ all the $g^{(i)}(z, \varepsilon)$ will be uniformly bounded in ε. For $r = 2$ the corresponding system becomes inhomogeneous (the terms $a_{j1}^{\prime(i)}(z)$) are present in the right sides). Since the rank of the system matrix is three, as before, then the coefficients $a_{j2}^{(i)}(z)$ $(i = 2, 3, 4)$ can be expressed /10/ in terms of $a_{12}^{(i)}(z)$. In turn, the coefficient $a_{12}^{(i)}(z)$ is represented by virtue of its holomorphism for $z = h$ /10/ by the series

$$a_{12}^{(i)}(z) = \sum_{k=0}^{\infty} c_k^{(i)} z_k, \quad i = 1, 2, 3, 4 \tag{3.9}$$

Substituting (3.9) into (3.8) and collecting coefficients of identical powers, we obtain a homogeneous system to determine the $c_k^{(i)}$ which is similar to (3.8). Its determinant is also different from zero, from which it follows

$$c_k^{(i)} \equiv 0;\ i = 1, 2, 3, 4;\ k = 0, 1, 2, \ldots$$

For values $r > 2$ the discussion is analogous to that for the case $r = 0$ since the coefficients for ε^{r-1} are zero. We obtain

$$g_j(z, \varepsilon) \sim \mathrm{const}\cdot\varepsilon,\ \varepsilon \to 0$$

Then

$$u_3 \sim \mathrm{const}/\alpha,\ \alpha \to \infty.$$

4. To study the influence of the initial stress on the wave process excited in a prestressed layer, we took the following dependences of the elastic parameters and the initial stresses on the coordinates as an illustration:

$$\begin{gathered} \lambda(z) = \lambda_0 h(1+\alpha)/(z+\alpha h), \quad \mu(z) = \mu_0 h(1+\beta)/(z+\beta h) \\ \rho(z) = \rho_0 \exp[\gamma(h-z)], \qquad \sigma(z) = \sigma_0 \exp[\delta(h-z)] \end{gathered} \tag{4.1}$$

or

$$\begin{gathered} \lambda(z) = \lambda_0 \exp[\alpha(h-z)], \quad \mu(z) = \mu_0 \exp[\beta(h-z)] \\ \rho(z) = \rho_0 \exp[\gamma(h-z)], \qquad \sigma(z) = \sigma_0 h(1+\delta)/(z+\delta h) \end{gathered} \tag{4.2}$$

An analysis showed the absence of qualitative distinctions in the pattern of zero and pole distributions for the function $K^\circ(\alpha)$ as the initial stresses and the elastic characteristics of the medium varied according to the laws (4.1) and (4.2), and the pattern presented in /1,2/. Only certain quantitative changes were observed. Here and below, the computations were carried out for the values $\lambda_0 = 926\cdot10^5$ N/m^2, $\mu_0 = 775\cdot10^5$ N/m^2, $\rho_0 = 10^3$ kg/m^3, $\sigma_0 = 10^{-3}\mu_0$, $\alpha = \delta = 0.5$ and $\beta = \gamma = 1$.

For all values of ω strict alternation of the zeros and poles holds. In combination with the properties noted above, the latter circumstance permits the conclusion /8/ that the problem of stamp vibration on a layer surface is solvable single-valuedly for any right sides of (2.7) in the class of functions that are continuous with weight in $[-a, a]$.

Let us approximate the function $K^\circ(u)$ by the function /8/

$$K^*(u) = c_1(u^2+B^2)^{-1/2} \prod_{k=1}^{n} (u^2-\zeta_k^2)(u^2-\gamma_k^2)^{-1} \tag{4.3}$$

Here c_1 and B are approximation parameters, γ_k $(k = 1, 2, \ldots, m)$, ζ_k $(k = 1, 2, \ldots, m_1)$ are real zeros and poles of the function $K^\circ(u)$, the remaining γ_k $(k = m+1, \ldots, n)$ and ζ_k $(k = m_1+1, \ldots, n)$ are determined from the condition of the best approximation.

The form of the functions $q(x_1), |x_1| \leqslant$ and $u_3(x_1), |x_1| \geqslant a$ determining the constant stress distribution under the stamp and the behavior of the free surface of the medium in the case $u_3(x_1) = \exp(i\eta x_1)$ $(|x_1| \leqslant a)$ and the approximating function (4.3) in different forms can be found in /8,11–13/.

Let us use the notation

$$\eta = \begin{cases} \mu_0^{-1}\operatorname{Re} q_0, & \sigma_0 = 0 \\ \mu_0^{-1}(\operatorname{Re} q_0 - \operatorname{Re} q_\sigma) \times 100, & \sigma_0 \neq 0 \end{cases}$$

$$\theta = \begin{cases} \operatorname{Re} u_{30}, & \sigma_0 = 0 \\ (\operatorname{Re} u_{30} - \operatorname{Re} u_{3\sigma}) \times 20, & \sigma_0 \neq 0 \end{cases}$$

Graphs of the functions η and θ are presented in Figs.1 and 2 for $\sigma_0 = 0,\ 5\cdot10^{-4}\mu_0$ and $5\cdot10^{-3}\mu_0$ (curves *1,2,3*, respectively) when the elastic parameters λ, μ, ρ and σ are described by (4.1) (solid curves) or (4.2) (dashes).

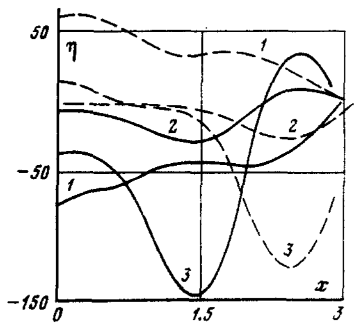

Fig.1

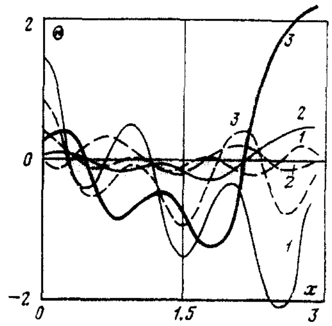

Fig.2

Computations were performed for the above-mentioned values of the parameters $\lambda_0, \mu_0, \rho_0, \alpha, \beta, \eta, \gamma, \delta$. The stamp half-width is $a = 3$, $\omega = 5\cdot10^4$ Hz.

The numerical analysis shows that a change in initial stress intensity exerts substantial influence on the contact stress distribution and the free surface displacement, where it is strongest at inflection points of the functions $q(x_1)$ and $u_3(x_1)$. For different laws of initial stress and elastic parameter variation for the medium the contact stresses and the free surface displacement are distinct; however, the nature of the initial stress influence remains qualitatively as before.

5. The problem of the radial vibration of a rigid $2a$ wide belt on a cylinder surface $r \leqslant R, |z| \leqslant \infty$ with initial stresses and elastic characteristics of the material varying along the radius is described by a system of equations with the boundary conditions

$$(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2} + \left(\lambda' + 2\mu' + \frac{2(\lambda+\mu)}{r}\right)\frac{\partial u_r}{\partial r} + (\lambda + a_{31})\frac{\partial^2 u_z}{\partial r\,\partial z} + \tag{5.1}$$

$$\left(\lambda' + \frac{\lambda}{r}\right)\frac{\partial u_z}{\partial r} + b_{31}\frac{\partial^2 u_r}{\partial z^2} + \frac{\lambda'}{r}u_r = -\rho\omega^2 u_r$$

$$b_{13}\frac{\partial^2 u_z}{\partial r^2} + (\lambda + a_{13})\frac{\partial^2 u_r}{\partial z\,\partial r} + \left(a_{13}' + \frac{a_{13}}{r} + \frac{\lambda}{r}\right)\frac{\partial u_r}{\partial z} +$$

$$\left(b_{13}' + \frac{b_{13}}{r}\right)\frac{\partial u_z}{\partial r} + (\lambda+2\mu)\frac{\partial^2 u_z}{\partial z^2} = -\rho\omega^2 u_z$$

$$r = R,\quad (\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{u_r}{r} + \lambda\frac{\partial u_z}{\partial z} = q_r(z),\quad |z| \leqslant a \tag{5.2}$$

$$a_{13}\frac{\partial u_r}{\partial z} + b_{13}\frac{\partial u_z}{\partial r} = 0$$

$$r = 0,\ u_r = 0,\ \tau_{rz} = 0,\ |z| \leqslant \infty$$

(u_r, u_z are the radial and axial displacements).

Applying the Fourier transformation in z and introducing the notation

$$V_r' = y_1,\ i\alpha V_z' = y_2,\ V_r = y_3,\ i\alpha V_z = y_4 \tag{5.3}$$

(V_r, V_z are the transforms of the functions u_r and u_z), we arrive at a system of four first-order differential equations of the type (2.1) and (2.2) from the system (5.1). Further constructions do not differ qualitatively from those in Sects.2–4. The integral equation of type (2.7) obtained was investigated by numerical methods, its solvability in the class of functions mentioned in Sect.4 was established. Using (4.3) as the approximating kernel of the integral equation, we obtain the possibility of using the results in /8,12/ to determine the form of the functions $q_r(z), |z| \leqslant a$ and $u_r(z), |z| > a$ that yield the distribution of the contact stresses under the belt and the behavior of the free surface.

Even in this case changes in the initial stresses influence the wave process in the cylinder substantially. Utilization of different laws also results in significant changes in the quantitative characteristics of this process.

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